1 Rayleigh-Benard Convection

Chandrasekhar[1] has carefully studied linear stability of Rayleigh-Benard convection using the Boussinesq approximation, given in terms of the Rayleigh (Ra) and Prandtl (Pr) numbers as

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} = -\nabla p + \frac{1}{Pr}\nabla^2\mathbf{u} - RaPrT, \qquad \frac{\partial T}{\partial t} + \mathbf{u}\cdot\nabla T = \nabla^2 T,$$

along with continuity, $\nabla\cdot\mathbf{u}$=0. Above a certain critical Rayleigh number Ra_c, the conduction due to adverse temperature gradient becomes linearly unstable to small perturbations and convection rolls develop (Fig. 1 top). The table below shows predictions of Ra_c at the most unstable wavenumber k_c for $\Omega = [0 : 2\pi/k_c] \times [0 : 1]$ having periodic conditions in x. Dirichlet conditions $T = 1 - y$ are specified for temperature on the horizontal boundaries, and three different conditions are considered for velocity: both walls (Dirichlet-Dirichlet), both stress-free (Neumann-Neumann) and a mix (Dirichlet-Neumann).

Critical Rayleigh number for 3 types of boundaries							
BC	Ra_{12}	Ra_{23}	Ra_c[4]	k_c[4]	Ra_1	Ra_2	Ra_3
D–D	1707.75	1707.74	1707.76	3.117	1760	1740	1725
D–N	1100.71	1100.64	1100.65	2.682	1144	1122	1111
N–N	657.639	657.566	657.511	2.2214	690	680	670

According to dynamical systems theory, the saturation amplitude (U) and kinetic energy ($\bar{E}_k$) grow, respectively, as $\sqrt{\epsilon}$ and ϵ, for $\epsilon := (Ra - Ra_c)/Ra_c \ll 1$. Thus Ra_c can be determined from a linear fit of (volume-averaged steady state) $\bar{E}_k$ versus Ra for two or more values of Ra as shown in Fig. 1 (left). In the table above, the estimates Ra_{12} and Ra_{23} are determined from solution pairs at (Ra_1, Ra_2) and (Ra_2, Ra_3), respectively. For each case, E_k is computed in a single unsteady run (Fig. 1 right) by varying Ra after time marching to a steady state such that $|dE_k/dt| \le \sigma \times \bar{E}_k$ with $\sigma = 10^{-4}$. Larger σ or lower polynomial orders $N < 7$ lead to a drop in accuracy for the estimates of Ra_c. Iteration tolerances were controlled by setting TOLREL $= 10^{-5}$. Larger values (10^{-3}) did not yield a clear initial linear stage with exponential growth but the Ra_c estimate was not affected.

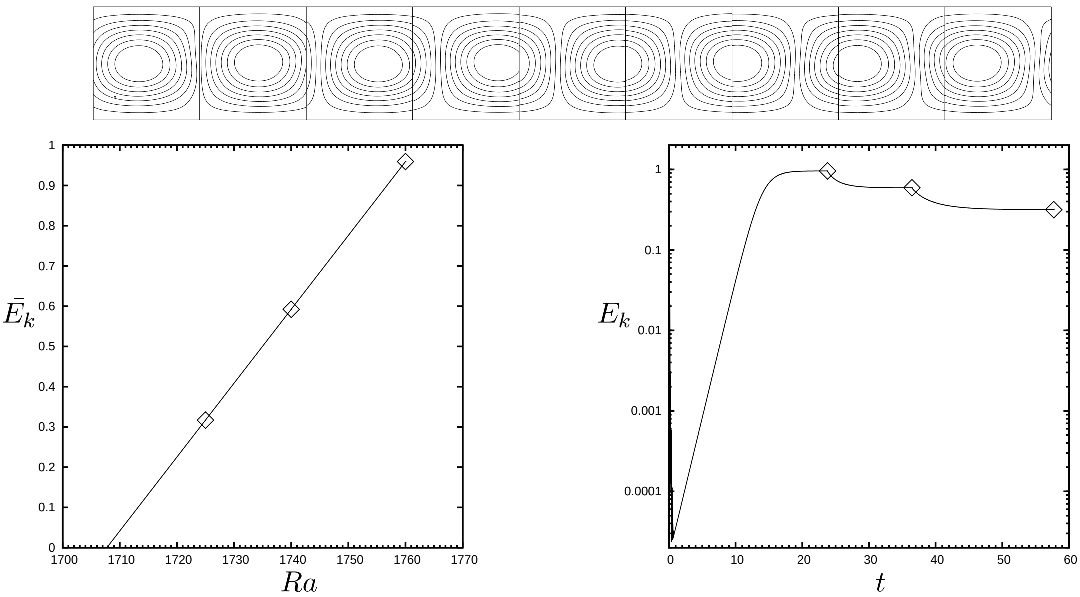

Figure 1: Rayleigh-Benard convection with walls: (top) streamlines for E=9 elements and N=7; (left) kinetic energy for E=3 and N=7 versus Rayleigh number and (right) time.

[1]S. Chandrasekhar, "Hydrodynamic and Hydromagnetic Stability," Oxford University Press (1961)

Rayleigh-B
enard Exercise

Consider the Rayleigh-B
enard (RB) problem of Section **??** for varying Prandtl numbers, $Pr := \nu/\alpha$. The flow is governed by the Navier-Stokes and energy equations, $\partial\mathbf{u}\frac{}{\partial t+\mathbf{u}\cdot\nabla\mathbf{u}=-\nabla p+\nu\nabla^2\mathbf{u}+\beta T\frac{\partial T}{\partial t}+\mathbf{u}\cdot\nabla T=-\nabla p+\alpha\nabla^2 T,(0)}$ which are coupled through the Boussinesq term βT. The domain is the rectangular box $\Omega = [0, L_x] \times [0, 1]$ with periodic boundary conditions in x and homogeneous Dirichlet conditions $\mathbf{u} = 0$ at y=0 and 1. Thermal boundary conditions are T=1 at y=0 and T=0 at y=1. Under sufficiently strong loading conditions, $\beta \gg 1$.

Explore Ra_c in the large- and small-Pr limits by varying Pr